Geometric Variational Problems Arising in Reaction-Diffusion Systems (Free Boundary Problems)

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Geometric Variational Problems Arising in Reaction-Diffusion Systems

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Reaction-diffusion systems have served as a pradigm to model various pattern formation phenomena in nature. Since almost all patterns observed in nature are usually recognized as an interface between two (or more) bulk states of differing properties, much attention has been given in recent years to the study of interface dynamics in reaction-diffusion systesm. In such a study, one usually derive systems of equations governing the dynamics of interfaces, called interface equations. There are many aspects in dealing with pattern formation phenomena in terms of interface equations. From a mathematical standpoint, the following are major ones:

(1) How to derive interface equations from the original reaction-diffusion systems
(2) To show the well-posedness of the interface equations thus obtained.
(3) To study the quantitative and qualitative behaviors of the solutions to the interface equations.
(4) To establish rigorous relationships between the results obtained in item (3) and the properties of solutions to the original reaction-diffusion systems.

In regard to items (1) and (4), one must keep in mind that there may be more than one set of interface equations, according to the temporal scale employed in the reaction-diffusion system.

Our purpose in this article is to show an example in which interface equations are derived from a geometric variational problem. We also show that non-degenerate equilibria of the interface equations give rise to an equilibrium solution of the original reaction-diffusion systems with the information on stability being inclusive. Schematically, our results may be described in the following way.

- Reaction-Diffusion System
→ Geometric Variational Problem
→ Find non-degenerate critical points
→ Equlibrium solutions to Reaction-Diffusion System.

In §1, we deal with scalar equations with spatially inhomogeneous reaction terms. The results in this section are equally valid for gradient systems. In §2, we generalize the results in §1 to non-gradient systems of reaction-diffusion equations with homogeneous reaction terms. In §3, we outline basic ideas in the proofs of main results.

1. Scalar Reaction-Diffusion Equations

Let us consider a spatially inhomogeneous reaction-diffusion equation

$$(1.1)\qquad \begin{cases} \epsilon^2 \dfrac{\partial u}{\partial t} = \epsilon^2 \Delta u - f(u,x,\epsilon) & \left(x \in \mathcal{D} \subset \mathbb{R}^N,\ t>0\right), \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & (x \in \partial\mathcal{D},\ t>0). \end{cases}$$

In (1.1), $\mathcal{D}$ is a smooth bounded domain and $\mathbf{n}$ stands for the inward unit normal vector on $\partial\mathcal{D}$. The nonlinear term $f(u,x,\epsilon)$ is assumed to be smooth and derived from a double-well potential $W(u,x,\epsilon)$:

$$(1.2)\qquad f(u,x,\epsilon) = \frac{\partial W(u,x,\epsilon)}{\partial u}$$

with $u = \phi^{(\pm)}(x,\epsilon)$ denoting the locations of two wells and $u = \phi^{(0)}(x,\epsilon)$ denoting the intermediate zero of f, satisfying

$$(1.2\text{-a})\qquad \phi^{(-)}(x,\epsilon) < \phi^{(0)}(x,\epsilon) < \phi^{(+)}(x,\epsilon) \qquad x \in \overline{\mathcal{D}}.$$

When the *layer parameter* $\epsilon > 0$ is small, it is known [2] that the solution of (1.1) with an initial condition in appropriate class develops internal layers in a short time and that the location of the layers (called interfaces) moves according to certain law of motion. In the latter stage of dynamical behavior, the difference in the values of potential at the two wells plays an important role. Let us denote the difference at each $x \in \overline{\mathcal{D}}$ by $I(x)$:

$$(1.3)\qquad \begin{aligned} I(x) :=& \int_{\phi^{(-)}(x)}^{\phi^{(+)}(x)} f(u,x,0)du \\ =& W(\phi^{(+)}(x),x,0) - W(\phi^{(-)}(x),x,0) \end{aligned}$$

where $\phi^{(\pm)}(x) = \phi^{(\pm)}(x,0)$ (cf. (1.2-a)).

Under the situation above, it is known that the boundary value problem

$$(1.4)\qquad \begin{cases} \dfrac{d^2\tilde{Q}_0}{d\tau^2} + c\dfrac{d\tilde{Q}_0}{d\tau} - f(\tilde{Q}_0,x,0) = 0 & \tau \in \mathbb{R}, \\ \lim_{\tau\to\pm\infty} \tilde{Q}_0(\tau) = \phi^{(\pm)}(x), \quad \tilde{Q}_0(0) = \phi^{(0)}(x) \end{cases}$$

has a unique solution $(\tilde{Q}_0(\tau;x), c(x))$, where $x \in \overline{\mathcal{D}}$ is regarded as a parameter. The potential difference $I(x)$ in (1.3) is related to the *local wave speed* $c(x)$ as follows.

$$(1.5)\qquad I(x) = c(x)\int_{-\infty}^{\infty} \left(\frac{\partial \tilde{Q}_0(\tau;x)}{\partial \tau}\right)^2 d\tau.$$

Let us rescale time in (1.1) such that the differential equation assumes the following form.

$$(1.1\text{-f})\qquad \epsilon\frac{\partial u}{\partial t} = \epsilon^2 \Delta u - f(u,x,\epsilon).$$

The interface equation for this problem is given by

$$(\text{IE-f})\qquad \mathbf{V}(x;\Gamma(t)) = c(x) \qquad x \in \Gamma(t),\quad t>0,$$

where $\mathbf{V}(x;\Gamma(t))$ stands for the normal velocity of $\Gamma(t)$. For a given interface Γ, we denote by $\mathcal{D}_\Gamma^{(-)}$ and $\mathcal{D}_\Gamma^{(+)}$ two components of $\mathcal{D}\backslash\Gamma$, and let $\nu(x;\Gamma)$ stand for the unit normal vecotor on Γ pointing into $\mathcal{D}_\Gamma^{(+)}$ (cf. Figure 1). The normal velocity $\mathbf{V}(x;\Gamma(t))$ is always measured along $\nu(x;\Gamma(t))$. Here and in what follows, we always treat the cases

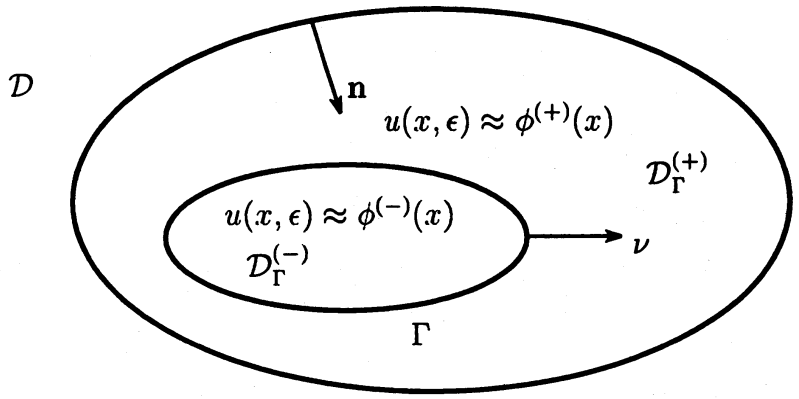

FIGURE 1. Γ divides $\mathcal{D}$ into two parts, $\mathcal{D}_\Gamma^{(-)}$ and $\mathcal{D}_\Gamma^{(+)}$.

where interfaces stay uniformly away from the boundary $\partial\mathcal{D}$ of doamin.

From the standpoint of investigating the existence of equilibrium internal layer solutions, it is natural to ask the next question:

> If the interface equation (IE-f) has a smooth equilibrium solution Γ, then does (1.1) have a family of equilibrium solutions with transition layers on Γ for small $\epsilon > 0$?

It turns out that the answer to this question is rather delicate. In [3], Fifé and Greenlee prove that the answer is affirmative if the condition

$$\nabla_x c(x)\Big|_\Gamma \cdot \nu(x,\Gamma) < 0 \qquad x \in \Gamma$$

is fulfilled, where Γ is a smooth equilibrium solution of (IE-f), namely, $\Gamma = \{\ x \in \mathcal{D} \mid c(x) = 0\ \}$ which, we suppose, is a closed manifold of codimension 1. Moreover, the solution thus obtained is a stable equilibrium of (1.1). The equilibrium solution $u(x,\epsilon)$ has the following behavior for each $d_0 > 0$:

$$\lim_{\epsilon\to 0} u(x,\epsilon) = \begin{cases} \phi^{(-)}(x) & x \in \overline{\mathcal{D}_\Gamma^{(-)}}\backslash\Gamma^{(d_0)} \\ \\ \phi^{(+)}(x) & x \in \overline{\mathcal{D}_\Gamma^{(+)}}\backslash\Gamma^{(d_0)} \end{cases} \qquad \text{uniformly,}$$

where $\Gamma^{(d_0)}$ stands for the d_0-neighborhood of Γ. It is of crucial importance to note that the normal vector ν above is pointing into the *region where the solution u assumes values close to $\phi^{(+)}$*. Since $c(x) \equiv 0$ on Γ, the condition above says that in the two regions away from the interface Γ the solution takes values close to absolute minima of the potential $W(u,x,0)$.

On the other hand, it is also pointed out in [12], in the context of the same question for a system of reaction-diffusion equations, that if, on the other hand, the condition

$$\nabla_x c(x)\Big|_\Gamma \cdot \nu(x,\Gamma) > 0 \qquad x \in \Gamma$$

is the case, then there may exist infinitely many internal layer solutions which exhibit sharp transitions near Γ. In radially symmetric cases, the validity of the latter statement has been established in [13]. By examining the proof in [12] and interpreting it in our situation, we can state the following criterion on the existence of equilibrium internal layer solutions.

Criterion 1: Let Γ be a smooth equilibrium solution of (IE-f). If it is non-degenerate in the sense that the spectrum of the linearized operator

$$L^\epsilon\varphi := \epsilon\left(\Delta^\Gamma + \sum_{j=1}^{N-1}\kappa_j(x)^2\right)\varphi + \left(\nabla_x c(x)\Big|_\Gamma \cdot \nu(x,\Gamma)\right)\varphi \qquad x \in \Gamma,$$

defined on Γ, is bounded away from zero uniformly in $\epsilon \in (0, \epsilon_0]$ for some $\epsilon_0 > 0$, then (1.1) has a family of solutions with internal transition layer on Γ. In the above, Δ^Γ is the Laplace-Beltrami operator on Γ and κ_i $(j = 1, \ldots, N-1)$ stand for principal curvatures of Γ.

Since Δ^Γ is a non-positive oprator, it is apparent that if $\nabla_x c(x)\Big|_\Gamma \cdot \nu(x,\Gamma) < 0$ then the spectrum of the operator L^ϵ is bounded away from zero uniformly in $\epsilon > 0$. Hence the criterion above is compatible with the result given by Fife and Greenlee [3].

Let us consider the following situation.

A1 $I(x) \equiv 0$ on $\overline{\mathcal{D}}$, or equivalently, $c(x) \equiv 0$ on $\overline{\mathcal{D}}$.

If **A1** is the case, there arises two kinds of degeneracy:

(1) *Any closed manifold* $\Gamma \subset \mathcal{D}$ of codimension one is an equilibrium of (IE-f)

(2) The corresponding linear operator L^ϵ in the **Criterion 1** reduces to ϵ times the Jacobi-operator on Γ and hence it has many small eigenvalues converging to 0 as $\epsilon \to 0$, making the criterion above powerless.

Therefore, we need first to establish a selection principle to identify possible equilibrium interfaces.

Along the line of arguments employed in Nakamura et al. [6], one can show that the interface equation for (1.1) is given by

$$\text{(1.6)} \qquad \mathbf{V}(x;\Gamma(t)) = -\kappa(x;\Gamma(t)) - \frac{\nabla_{\nu_\Gamma} m(x)}{m(x)} + \frac{\alpha(x)}{m(x)} \qquad x \in \Gamma(t), \quad t > 0,$$

where $\kappa(x;\Gamma)$ stands for the sum of principal curvature of Γ and

(1.6-a)

$$m(x) = \int_{-\infty}^{\infty}\left(\frac{\partial \tilde{Q}_0(\tau;x)}{\partial\tau}\right)^2 d\tau \quad x \in \overline{\mathcal{D}} \qquad \text{(unit transition momentum at } x\text{)},$$

(1.6-b)

$$\alpha(x) = \int_{-\infty}^{\infty} f_\epsilon(\tilde{Q}_0(\tau;x), x, 0)\frac{\partial \tilde{Q}_0(\tau;x)}{\partial\tau} d\tau \quad x \in \overline{\mathcal{D}} \qquad \text{(excess-energy of order } \epsilon\text{)}.$$

We now assume that the following conditions are fulfilled.

A2 The interface equation (1.6) has a smooth equilibrium solution $\Gamma \subset \mathcal{D}$.

A3 The equilibrium Γ is non-degenerate in the sense that the linear operator $\mathcal{A}$ defined below does not have 0 as its eigenvalue:

$$\begin{aligned}(1.7)\qquad \mathcal{A}\varphi(x) :=& m(x)\left(\Delta^{\Gamma} + \sum_{j=1}^{N-1}\kappa_j(x)^2\right)\varphi(x) + \nabla_{\Gamma} m(x)\cdot\nabla_{\Gamma}\varphi(x) \\ &+ \Big(-\kappa(x;\Gamma)\nabla_{\nu_\Gamma} m(x) - \nabla_{\nu_\Gamma}^2 m(x) + \nabla_{\nu_\Gamma}\alpha(x)\Big)\varphi(x) \qquad x\in\Gamma,\end{aligned}$$

where ∇_{Γ} is the gradient operator on Γ.

We have:

Lemma 1.1. *The operator $\mathcal{A}$ is self-adjoint and its eigenvalues are all real:*

$$(1.8)\qquad \sigma(\mathcal{A}) = \{\,\lambda_j\,\}_{j=0}^{\infty} \subset \mathbb{R}, \qquad \lambda_0 > \lambda_1 > \ldots > \lambda_j \to -\infty,$$

where only distinct eigenvalues *are listed. The* multiplicity *of λ_j is denoted by $m_j \geq 1$.*

Let us now define a functional $F(\Gamma)$ by

$$(1.9)\qquad F(\Gamma) := \int_{\Gamma} m(x)dS_x^{\Gamma} - \int_{\mathcal{D}_{\Gamma}^{(-)}} \alpha(x)dx \qquad (\partial\mathcal{D}_{\Gamma} = \Gamma,$$

where dS_x^{Γ} stands for the *surface element* on Γ.

Lemma 1.2. *The Euler-Lagrange equation for F is given by*

$$(1.10)\qquad -\kappa(x;\Gamma)m(x) - \nabla_{\nu_\Gamma} m(x) + \alpha(x) = 0 \qquad x\in\Gamma,$$

and the second variation of F is described by $\mathcal{A}\varphi$ defined in (1.7).

Note that (1.10) is the equation for equilibrium solutions of the interface equation (1.6). Our main result is

Theorem 1.3 ([7]). *Under the conditions* **A1**, **A2**, *and* **A3**, *there exist $\epsilon_0 > 0$ and a family of equilibrium solutions $u(x,\epsilon)$ of* (1.1), *defined for $\epsilon \in (0,\epsilon_0]$, such that for each $d_0 > 0$ fixed*

$$(1.11)\qquad \lim_{\epsilon\to 0} u(x,\epsilon) = \begin{cases} \phi^{(-)}(x) & x\in \overline{\mathcal{D}_{\Gamma}^{(-)}}\backslash\Gamma^{(d_0)} \\ \\ \phi^{(+)}(x) & x\in \overline{\mathcal{D}_{\Gamma}^{(+)}}\backslash\Gamma^{(d_0)} \end{cases} \qquad \textit{uniformly},$$

where $\mathcal{D}_{\Gamma}^{(\pm)}$ are two regions ($\subset\mathcal{D}$) separated by Γ and $\Gamma^{(d_0)}$ stands for the d_0-neighborhood of Γ.

Moreover, if $\lambda_0 < 0$ then $u(x,\epsilon)$ is asymptotically stable, and if $\lambda_k > 0 > \lambda_{k+1}$ for some integer $k \geq 0$ then $u(x,\epsilon)$is unstable with instability index equal to $\sum_{j=0}^{k} m_j$.

Conclusion. Non-degenerate critical points of the functional F in (1.9), if regular enough, give rise to equilibrium solutions of (1.1). The *index* of the critical point is equal to the *dimension of the unstable manifold* of the equilibrium solution. It is an amusing fact that the formulae (1.6) and (1.7) naturally appear in matched asymptotic expansions.

The proof of Theorem 1.3 depends on the methods developed in [13, 15] (matched asymptotic expansions). When $m(x) \equiv 1$ and $\alpha(x) \equiv 0$, the same result as Theorem 1.3 was first obtained by [5] for stable case, by using Γ-convergence and related variational techniques. Our theorem extends those in [5] to cover unstable cases. Theorem 1.3 prompts the resolution of the following.

Geometric Variational Problem 1.
Find critical points of the functional F in (1.9).

We call this problem *geometric* because the unknown Γ is a geometric object.

2. System of Reaction-Diffusion Equations

We now move on to deal with reaction-diffusion *systems.*

$$\text{(2.1)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \epsilon \Delta u - \dfrac{1}{\epsilon} f(u, v, \epsilon) & \\ & (x \in \mathcal{D},\ t > 0), \\ \dfrac{\partial v}{\partial t} = D \Delta v + g(u, v, \epsilon) & \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 = \dfrac{\partial v}{\partial \mathbf{n}} & (x \in \partial \mathcal{D},\ t > 0). \end{cases}$$

The first equation in (2.1) looks almost identical to (1.1), if we replace x in the latter by $v(x)$. We assume in this section that the nonlinear term $f(u, x, \epsilon)$ is smooth and derived from a double-well potential $W(u, v, \epsilon)$:

$$\text{(2.2)} \qquad f(u, v, \epsilon) = \frac{\partial W(u, v, \epsilon)}{\partial u}$$

with $u = h^{(\pm)}(v, \epsilon)$ denoting the locations of two wells, while $u = h^{(0)}(v, \epsilon)$ stands the intermediate zero of $f(\cdot, v, \epsilon)$, satisfying

$$\text{(2.2-a)} \qquad h^{(-)}(v, \epsilon) < h^{(0)}(v, \epsilon) < h^{(+)}(v, \epsilon) \qquad v \in \mathbb{R}.$$

Similar to scalar case, the difference in the values of potential at the two wells will play an important role in describing the dynamics of (2.1). Let us denote the difference at each v by $J(v)$:

$$\text{(2.3)} \qquad \begin{aligned} J(v) &:= \int_{h^{(-)}(v)}^{h^{(+)}(v)} f(u, v, 0) du \\ &= W(h^{(+)}(v), v, 0) - W(h^{(-)}(v), v, 0) \end{aligned}$$

where $h^{(\pm)}(v) = h^{(\pm)}(v, 0)$ (cf. (2.2-a)).

It is known [2] that the solution of (2.1) with appropriate initial conditions develops transition layers in short time, and that the interface evolves according to the following

system of interface equations:

$$\text{(IE-a)} \qquad \mathbf{V}(x;\Gamma(t)) = c(v(x,t)) \qquad (x \in \Gamma(t),\ t>0),$$

$$\text{(IE-b)} \qquad v_t = D\Delta v + g^*(v,x;\Gamma(t)) \qquad (x \in \mathcal{D}\backslash\Gamma(t),\ t>0),$$

$$\text{(IE-c)} \qquad \partial v(x,t)/\partial \mathbf{n} = 0 \qquad (x \in \partial\mathcal{D},\ t>0),$$

$$\text{(IE-d)} \qquad \Gamma(0)=\Gamma_0,\quad v(x,0)=v_0(x) \qquad (x \in \mathcal{D}),$$

$$\text{(IE-e)} \qquad v(\cdot,t) \in C^1(\overline{\mathcal{D}}) \cap C^2(\mathcal{D}\backslash\Gamma(t)).$$

The function g^* in (IE-b) is defined by

$$g^*(v,x;\Gamma(t)) = \begin{cases} g^-(v) := g(h^{(-)}(v),v) & \text{if } x \in \mathcal{D}^{(-)}_{\Gamma(t)} \\ g^+(v) := g(h^{(+)}(v),v) & \text{if } x \in \mathcal{D}^{(+)}_{\Gamma(t)}. \end{cases}$$

The condidion in (IE-e) is called a C^1-matching condition.

It is also known [2] that the problem (IE) is well-posed and that its solutions do approximate the motion of the internal layer solutions of (2.1) on *finte* time intervals $[0,T_\epsilon]$ (although $T_\epsilon \to \infty$ as $\epsilon \to 0$). Since for $\epsilon > 0$ the approximation is valid only on finite time intervals in general, some of asymptotic information on the solutions of (2.1) may not be captured by only analysing the behavior of solutions of (IE-a,b,c,d). For example, the results in [2] do not answer the following question:

> *If* (IE-a,b,c,d) *has an equilibrium solution* $(\Gamma_0, v(x;\Gamma_0))$, *then, does* (2.1) *have a corresponding equilibrium solutions for small* $\epsilon > 0$*?*

The equilibrium solutions of (IE-a,b,c,d) have to satisfy

$$0 = c(v(x;\Gamma_0)) \qquad (x \in \Gamma_0),$$

$$0 = D\Delta v + g^*(v,x;\Gamma_0) \qquad (x \in \mathcal{D}\backslash\Gamma_0),$$

$$\partial v(x,\ \Gamma_0)/\partial \mathbf{n} = 0 \qquad (x \in \partial\mathcal{D})$$

$$v(\cdot;\Gamma_0) \in C^1(\overline{\mathcal{D}}) \cap C^2(\mathcal{D}\backslash\Gamma_0).$$

Let us denote by v^* a zero $c(v)$: $c(v^*)=0$. Therefore, they are a solution of the following *free boundary problem*:

$$\text{(FB-a)} \qquad 0 = D\Delta v + g^*(v,x;\Gamma_0) \qquad (x \in \mathcal{D}\backslash\Gamma_0),$$

$$\text{(FB-b)} \qquad v(x;\Gamma_0) = v^* \ \text{ on } \Gamma_0, \qquad \partial v(x;\Gamma_0)/\partial \mathbf{n} = 0 \ \text{ on } \partial\mathcal{D},$$

$$\text{(FB-c)} \qquad v(\cdot;\Gamma_0) \in C^1(\overline{\mathcal{D}}) \cap C^2(\mathcal{D}\backslash\Gamma_0).$$

Note that the nonlinearity $g^*(v,x;\Gamma_0)$, in general, has a jump discontinuity along Γ_0. The free boundary problem can be reformulated as:

Geometric Variational Problem 2.

Find critical points $v(\cdot;\Gamma)$ of the functional $\mathcal{F}(v;\Gamma)$:

$$\mathcal{F}(v;\Gamma) := \int_{\mathcal{D}} \left(\frac{1}{2} D|\nabla v|^2 - G^*(v,x;\Gamma) \right) dx$$

$$\left(\text{with } G^*(v,x;\Gamma) := \int_{v^*}^{v} g^*(s,x;\Gamma)ds \right).$$

Then identify Γ_0 so that $v(x;\Gamma_0) \equiv v^*$ on Γ_0.

For the sake of argument, let us assume that the free boundary problem (FB-a,b,c) has a regular solution $(V^*(x), \Gamma_0)$. Then it was shown in [12] that the *linearized eigenvalue problem* defined for $p(x)$ $(x \in \Gamma_0)$ and $q(x)$ $(x \in \mathcal{D})$:

$$\text{(EVP-1)} \qquad \lambda p = \epsilon \left(\Delta^{\Gamma_0} + \sum_{j=1}^{N-1} \kappa_j(x)^2 \right) p + c'(v^*) \frac{\partial V^*(x)}{\partial \nu(x)} \Big|_{\Gamma_0} p + c'(v^*) q|_{\Gamma_0} \qquad x \in \Gamma_0,$$

$$\text{(EVP-2)} \qquad \lambda q = D\Delta q + g_v^*(V^*(x), x; \Gamma_0) q - [g^*] p \otimes \delta_{\Gamma_0} \qquad x \in \mathcal{D}$$

plays an important role. In (EVP-1), $c'(v^*)$ is the derivative at $v = v^*$ of $c(v)$ with respect to v.

In (EVP-2), $[g^*]$ stands for the jump of g^* on Γ_0:

$$[g^*] = g(h^{(+)}(v^*), v^*) - g(h^{(-)}(v^*), v^*),$$

and the symbol δ_{Γ_0} stands for the Dirac-delta function supported on Γ_0. Therefore (EVP-2) should be interpreted in distributional sense. By writing it in weak form and integrating by parts, one can recast (EVP-2) as concisely as

$$\text{(EVP-2')} \qquad \Pi_\lambda q|_{\Gamma_0} + [g^*] p = 0 \qquad x \in \Gamma_0,$$

where Π_λ is the Dirichlet-to-Neumann map defined by

$$\Pi_\lambda q(x) := \Pi_\lambda^- q(x) + \Pi_\lambda^+ q(x) := \frac{\partial v_\lambda^-(x)}{\partial \nu(x)} - \frac{\partial v_\lambda^+(x)}{\partial \nu(x)} \qquad (x \in \Gamma_0)$$

in which $v_\lambda^\pm(x)$ are solutions of the following problem:

$$\begin{aligned} D\Delta v^\pm + g_v^*(V^*(x), x; \Gamma_0) v^\pm &= \lambda v^\pm \qquad (x \in \mathcal{D}_{\Gamma_0}^{(\pm)}, \quad \mathcal{D}_{\Gamma_0}^{(-)} \cup \mathcal{D}_{\Gamma_0}^{(+)} = \mathcal{D} \backslash \Gamma_0), \\ v^\pm(x) &= q(x) \qquad (x \in \Gamma_0), \\ \frac{\partial v^\pm(x)}{\partial \mathbf{n}} &= 0 \qquad (x \in \partial\mathcal{D}). \end{aligned}$$

Under the condition $g_v^* < 0$, one can show that Π_λ is invertible for $\mathrm{Re}\, \lambda \geq -\lambda_0$ for some $\lambda_0 > 0$.

The following criterion has been established in [12].

Criterion 2 If the linear operator $\mathcal{A}^\epsilon$, defined by

$$\text{(L)} \qquad \mathcal{A}^\epsilon p := \epsilon \left(\Delta^{\Gamma_0} + \sum_{j=1}^{N-1} \kappa_j(x)^2 \right) p + c'(v^*) \frac{\partial V^*(x)}{\partial \nu(x)} \Big|_{\Gamma_0} p - c'(v^*) \Pi_0^{-1} p \qquad x \in \Gamma_0,$$

is invertible uniformly in $\epsilon > 0$, then (2.1) has a family of equilibrium solutions exhibiting transition layers on Γ_0. Note that this operator $\mathcal{A}^\epsilon$ is obtained by substituting (EVP-2') with $\lambda = 0$ into (EVP-1).

When $c'(v^*) < 0$, it is in fact shown in [12] that the eigenvalues of (L) are bounded away from 0 uniformly in $\epsilon > 0$. If, on the other hand, $c'(v^*) > 0$ is the case, then the opeartor $\mathcal{A}^\epsilon$ has many small eigenvalues conveging to zero as $\epsilon \to 0$.

We denote by $(\tilde{Q}_0(\tau; v), c(v))$ the solution of (1.4) with x, $\phi^{(\pm)}$, and $\phi^{(0)}$ begin replaced by v, $h^{(\pm)}(v)$, and $h^{(0)}(v)$. Similar to (1.5), $c(v)$ and $J(v)$ are related as

$$J(v) := c(v) \int_{-\infty}^{\infty} \left(\frac{\partial \tilde{Q}_0(\tau; v)}{\partial \tau} \right)^2 d\tau.$$

Let us now consider the following situation:

B1 $J(v) \equiv 0$ for $v \in \mathbb{R}$, or equivalently, $c(v) \equiv$ for $v \in \mathbb{R}$.

Under the condition **B1**, the first equation (IE-a) decouples from others and the interface equations reduce to

(IE-a') $\Gamma(t) \equiv \Gamma_0 \qquad (t \geq 0),$

(IE-b) $v_t = D\Delta v + g^{\ast}(v, x; \Gamma) \qquad (x \in \mathcal{D}\backslash\Gamma,\ t > 0),$

(IE-c) $\partial v(x,t)/\partial \mathbf{n} = 0 \qquad (x \in \partial\mathcal{D}),$

(IE-d') $v(x,0) = v_0(x) \qquad (x \in \mathcal{D}),$

(IE-e) $v(\cdot, t) \in C^1(\overline{\mathcal{D}}) \cap C^2(\mathcal{D}\backslash\Gamma).$

This equation is a gradient system associated with the potential

$$E(v) \equiv \int_{\mathcal{D}} \left(\frac{D}{2} |\nabla_x v(x)|^2 - G(v(x), x; \Gamma) \right) dx$$

$$\left(\text{with} \quad G(v, x; \Gamma) := \int_0^v g^{\ast}(s, x; \Gamma) ds \right),$$

and hence its solutions converge to an equilibrium solution as $t \to \infty$.

In order to state our problem succinctly, let us define a set $\mathcal{S}$ of interfaces.

(2.4) $\mathcal{S} = \left\{ \Gamma \subset \mathcal{D} \mid \Gamma \text{ is an } N-1 \text{ dimensional, smooth, } \textit{connected}, \text{ closed manifold} \right\}.$

Lemma 2.1. *Under the condition* **B1**, *assume that*

$$\frac{d}{dv} g^{\pm}(v) < 0 \qquad v \in \mathbb{R}.$$

For each $\Gamma \in \mathcal{S}$, *the problem*

(BVP-1)
$$\begin{cases} 0 = D\Delta v + g^{\ast}(v, x; \Gamma) & (x \in \mathcal{D}\backslash\Gamma) \\ \partial v(x)/\partial \mathbf{n} = 0 & (x \in \partial\mathcal{D}) \end{cases},$$

(BVP-2) $v(\cdot) \in C^1(\overline{\mathcal{D}}) \cap C^2(\mathcal{D}\backslash\Gamma).$

has a unique *solution* $v_\Gamma(x)$.

We encounter again a degenerate situation.

(1) Lemma 2.1 says that under the condition **B1** the interface equation (IE-a,b,c,d) has a continuum of equilibrium solutions $\{ v_\Gamma \mid \Gamma \in \mathcal{S} \}$.

(2) Moreover, for each member v_Γ of the family, the operator $\mathcal{A}^\epsilon$ does not satisfy the requirement in **Criterion 2**.

Therefore, under the condition **B1**, the interface equations (IE-a,b,c,d) do not capture essential dynamics of (2.1). To find a refined set of interface equations, we rescale time and consider (2.1) in the following version:

$$(2.5)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \dfrac{1}{\epsilon^2} f(u,v,\epsilon) \\ \dfrac{\partial v}{\partial t} = \dfrac{1}{\epsilon}\Big(D\Delta v + g(u,v,\epsilon)\Big) \end{cases} \qquad (x \in \mathcal{D},\ t>0)$$

with the same boundary conditions as in (2.1).

Following the procedures employed in Nakamura et al. [6], one can show that the interface equation for (2.5) is given by

$$(2.6)\qquad \mathbf{V}(x;\Gamma(t)) = -\kappa(x;\Gamma(t)) - \frac{\nabla_{\nu_\Gamma} m(v_{\Gamma(t)})}{m(v_{\Gamma(t)})} + \frac{\alpha(v_{\Gamma(t)})}{m(v_{\Gamma(t)})} \qquad x \in \Gamma(t),\quad t>0,$$

where

$$(2.6\text{-a})\quad m(v) = \int_{-\infty}^{\infty} \left(\frac{\partial \tilde{Q}_0(\tau;v)}{\partial \tau}\right)^2 d\tau \quad v \in \mathbb{R} \qquad \text{(unit transition momentum at } v),$$

$$(2.6\text{-b})\quad \alpha(v) = \int_{h^{(-)}(v)}^{h^{(+)}(v)} f_\epsilon(u,v,0)du \quad v \in \mathbb{R} \qquad \text{(excess-energy of order } \epsilon).$$

The well-posedness of (2.6) has been established in [1].

We now assume that the following conditions are fulfilled.

B2 The interface equation (2.6) has a smooth equilibrium solution $\Gamma \in \mathcal{S}$.

B3 The equilibrium Γ is non-degenerate in the sense that the linear operator $\mathcal{B}$ defined below does not have 0 as its eigenvalue:

$$\begin{aligned}(2.7)\qquad \mathcal{B}\varphi(x) :=& m(v_\Gamma)\left(\Delta^\Gamma + \sum_{j=1}^{N-1}\kappa_j(x)^2\right)\varphi(x) + \nabla_\Gamma m(v_\Gamma)\cdot\nabla_\Gamma\varphi(x) \\ &+ \Big(m'(v_\Gamma)\Delta^\Gamma v_\Gamma + \frac{g^+(v_\Gamma)+g^-(v_\Gamma)}{2D} - m''(v_\Gamma)|\nabla_{\nu_\Gamma} v_\Gamma|^2 + \nabla_{\nu_\Gamma}\alpha(v_\Gamma)\Big)\varphi \\ &+ \frac{[g^\kappa]}{D}\Big(-\kappa(x;\Gamma)m'(v_\Gamma) - m''(v_\Gamma)\nabla_{\nu_\Gamma} v_\Gamma + \nabla_{\nu_\Gamma}\alpha(v_\Gamma)\Big)\Pi_0^{-1}\varphi(x) \\ &+ \frac{[g^\kappa]}{D}\frac{m'(v_\Gamma)}{2}\Big(\Pi_0^- - \Pi_0^+\Big)\Pi_0^{-1}\varphi(x) \qquad x \in \Gamma,\end{aligned}$$

where

$$g^\pm(v) = g(h^{(\pm)}(v), v).$$

Although the operator $\mathcal{B}$ looks quite complicated, it enjoys the following property.

Lemma 2.2. *The operator $\mathcal{B}$ is self-adjoint and its eigenvalues are all real:*

$$(2.8)\qquad \sigma(\mathcal{B}) = \{\ \lambda_j^s\ \}_{j=0}^{\infty} \subset \mathbb{R}, \qquad \lambda_0^s > \lambda_1^s > \ldots > \lambda_j^s \to -\infty,$$

where only distinct eigenvalues *are listed. The* multiplicity *of λ_j^s is denoted by $m_j^s \geq 1$.*

Remark 2.3. The operator $\mathcal{B}$ is self-adjoint only when the interface $\Gamma \in \mathcal{S}$ is *connected*. If Γ has more than one connected components, the operator $\mathcal{B}$ may not be self-adjoint.

Let us now define a functional $F_s(\Gamma)$ by

$$F_s(\Gamma) := \int_\Gamma m(v_\Gamma) dS_x^\Gamma - \int_{\mathcal{D}_\Gamma^{(-)}} \alpha(v_\Gamma) dx, \tag{2.9}$$

where v_Γ is the solution given in Lemma 2.1.

Lemma 2.4. *The Euler-Lagrange equation for F_s is given by*

$$-\kappa(x;\Gamma) m(v_\Gamma) - \nabla_{\nu_\Gamma} m(v_\Gamma) + \alpha(v_\Gamma) = 0 \qquad x \in \Gamma, \tag{2.10}$$

and the second variation of F_s is described by $\mathcal{B}\varphi$ defined in (2.7).

Our main result for system (2.1) is:

Theorem 2.5. *Under the conditions* **B1**, **B2**, *and* **B3**, *there exist $\epsilon_0 > 0$ and a family of equilibrium solutions $(u(x,\epsilon), v(x,\epsilon))$ of* (2.1), *defined for $\epsilon \in (0, \epsilon_0]$, such that for each $d_0 > 0$ fixed*

$$\lim_{\epsilon \to 0} v(x,\epsilon) = v_\Gamma(x) \qquad \text{uniformly on } \overline{\mathcal{D}},$$

$$\lim_{\epsilon \to 0} u(x,\epsilon) = \begin{cases} h^{(-)}(v_\Gamma(x)) \\ \\ h^{(+)}(v_\Gamma(x)) \end{cases} \quad \text{uniformly on} \quad \begin{cases} \overline{\mathcal{D}_\Gamma^{(-)}} \backslash \Gamma^{(d_0)} \\ \\ \overline{\mathcal{D}_\Gamma^{(+)}} \backslash \Gamma^{(d_0)} \end{cases},$$

where $\mathcal{D}_\Gamma^{(\pm)}$ are two regions ($\subset \mathcal{D}$) separated by Γ and $\Gamma^{(d_0)}$ stands for the d_0-neighborhood of Γ.

Moreover, the eigenvalues of $\mathcal{B}$ determine the stability of the equilibrium solutions:

- *If $\lambda_0^s < 0$, then the solution is asymptotically stable.*
- *If $\lambda_k^s > 0 > \lambda_{k+1}^s$ for some integer $k \geq 0$, then the solution is unstable with instability index equal to $\sum_{j=0}^{k} m_j^s$.*

This theorem makes it meaningful to establish some methods to deal with the following problem.

Geometric Variational Problem 3.
Find critical points of the functional F_s in (2.9).

Remark 2.6. Even if the interface Γ has more than one connected components, the statement of Theorem 2.5 is still valid, except for the stability properties. In such a situation, the operator $\mathcal{B}$ is not self-adjoint and may have complex eigenvalues. Moreover, considering the diffusion coefficient D as a bifurcation parameter, one may be able to detect Hopf-bifurcations of interfaces. In fact, it is confirmed in [14] that the Hopf-bifurcation of interfaces can occur in the following system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \dfrac{1}{\epsilon^2} f(u,v,\epsilon) \\ \\ \dfrac{\partial v}{\partial t} = \dfrac{1}{\epsilon}\left(\dfrac{D}{\epsilon}\Delta v + g(u,v,\epsilon)\right) \end{cases} \qquad (x \in \mathcal{D},\ t > 0) \tag{2.11}$$

with $\langle v \rangle \not\equiv 0$ which is very similar to (2.5). The interface equations for (2.11) is given by

$$\begin{aligned} \mathbf{V}(x;\Gamma(t)) &= -\kappa(x;\Gamma(t)) + c'(v^*)v(x;\Gamma(t)) && (x \in \Gamma(t),\ t \geq 0) \\ -D\Delta v(x;\Gamma(t)) &= P(x;\Gamma(t)) && (x \in \mathcal{D}\backslash\Gamma(t),\ t \geq 0) \\ \partial v/\partial \mathbf{n} &= 0 && (x \in \partial\mathcal{D},\ t \geq 0) \\ \Gamma(0) = \Gamma_0, \quad v(x;\Gamma(0)) &= v_0(x) && (x \in \overline{\mathcal{D}}) \\ v(\cdot;\Gamma(t)) \in C^2(\overline{\mathcal{D}}\backslash\Gamma(t)) \cap C^1(\overline{\mathcal{D}}) && && (t \geq 0), \end{aligned}$$

where

$$P(x;\Gamma) = \begin{cases} g^-(v^*) & (x \in \mathcal{D}_\Gamma^{(-)}) \\ g^+(v^*) & (x \in \mathcal{D}_\Gamma^{(+)}) \end{cases}.$$

In a series of works [8, 9, 10, 11] on one dimensional reaction-diffusion systems, Nishiura and his co-workers have established a powerful method called the *Singular Limit Eigenvalue Problem* method (SLEP-method, for short) to determine the stability property of equilibrium transition layer solutions. The basic structure of the method is concisely expressed in the following diagram.

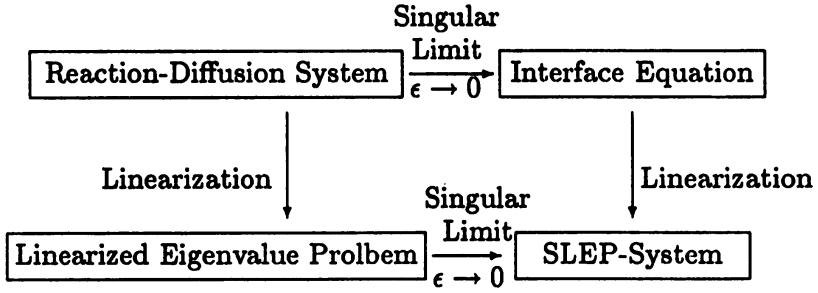

FIGURE 2. Relationship between reaction-diffusion system and SLEP-system.

First construct an equilibrium transition layer solution to the reaction-diffusion system and linearize the system around it to obtain an eigenvalue problem. The singular limit of the eigenvalue problem is called the SLEP-system which contains full information on the stability of the equilibrium. Moreover, Nishiura et al. show that the SLEP-system is also obtained by first passing to the singular limit of the reaction diffusion system to obtain an associated system of interface equations and then linearizing the latter around its equilibrium.

Our results fit pricisely into the same framework. We first find an equilibrium to the system of interface equations and linearize it to obtain a SLEP-system. Then our assertion is that if the SLEP-system thus obtained is non-degenerate, then the equilibrium of the system of interface equations gives rise to an equilibrium transition layer solution of the reaction-diffusion system. Moreover, the SLEP-system also carries the full information on the stability of the transition layer solution. We also point out the following facts which are guiding principles in our proof.

- The interface equations are nothing but the lowest order C^1-matching conditions.

- The SLEP-system is the principal part of the higer order C^1-matching conditions.

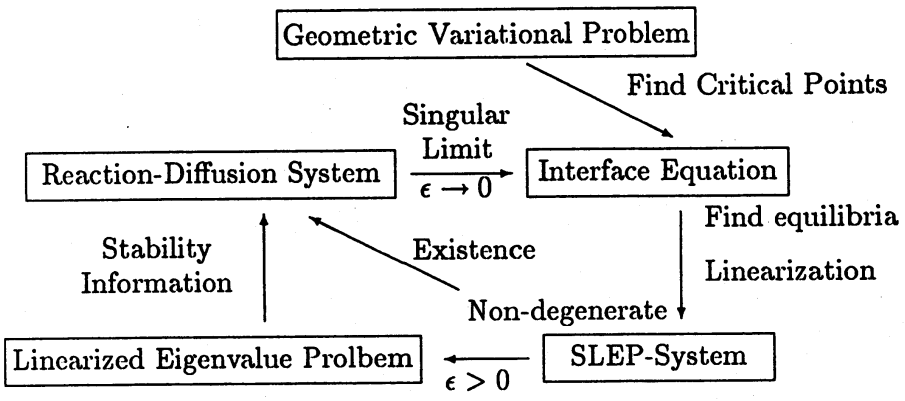

FIGURE 3. Non-degenerate equilibra of the interface equations give rise to transition layer solutions of reaction-diffusion system and their stability properties are completely determined by SLEP-system.

3. Outline of Proof

The proof consists of three steps:

(1) **Construction of highly accurate approximate solutions** U^ϵ_{app} via the method of matched asymptotic expansion. The conditions **A2** and **A3** (resp. **B2** and **B3**) allow us to find C^1-matched approximate solutions with arbitrarily high order of accuracy. As pointed out at the end of the previous section, **A2** (**B2**) is the lowest order C^1-matching condition, and **A3** (**B3**) allow us to find higher order C^1-matched approximations. Once an approximate solution is constructed, the original problem is then written as

$$\mathcal{L}^\epsilon \varphi + \mathcal{N}^\epsilon(\varphi) + \mathcal{R}^\epsilon = 0, \tag{3.1}$$

where $\mathcal{L}^\epsilon \varphi$ is obtained from the original problem by linearization around the approximate solution, $\mathcal{N}^\epsilon(\varphi)$ stands for nonlinear terms containing quadratic and higer order terms in φ, and $\mathcal{R}^\epsilon$ measures how well the approximation satisfies the original problem.

(2) **The spectral analysis** of $\mathcal{L}^\epsilon$. The linear operator $\mathcal{L}^\epsilon$ in general has small eigenvalues that go to zero as $\epsilon \to 0$, called critical eigenvalues. In the present situation, these critical eigenvalues are of order $O(\epsilon^2)$, and when divided by ϵ^2 they are essentially the eigenvalues of the SLEP-system. Therefore, **A3** (**B3**) guarantees that the linear operator $\mathcal{L}^\epsilon$ is invertible, although it has small eigenvalues that converge to zero as $\epsilon \to 0$.

(3) **To establish the solvability** of (3.1). Since the linear part $\mathcal{L}^\epsilon \varphi$ is small, $O(\epsilon^2)$, one needs to make the contribution of the nonlinear term $\mathcal{N}^\epsilon(\varphi)$ smaller than the linear part. This, in turn, is possible if the remainder term $\mathcal{R}^\epsilon$ is small enough, say, $\|\mathcal{R}^\epsilon\| = O(\epsilon^8)$ in the present situation. Thus one obtains the true solution U^ϵ very close to the approximate one. Now the linearization of the original problem around the genuine solution U^ϵ is a very small perturbation of $\mathcal{L}^\epsilon$, and hence the stability

properties of U^ϵ is completely determined by the SLEP-system which has already been analized in the previous Step (2).

The stragegy descirbed above seems to have a wide range of applicability in dealing with transition layers and interfaces. The same idea has been successfully applied in other situations [4, 15].

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